Concerning metrics of the Weyl type

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ABSTRACT

A novel look at the metric
$ds^2 = e^{2U}dt^2 - e^{2X}(dx^{1^2} + dx^{2^2}) - e^{2Z}dx^{3^2}$ where $X = \lambda + \upsilon - U$,
$Z = \mu - U$ is attempted. Physical components of the Riemann Christoffel tensor relative to a tetrad of unit vectors is found. It is hoped that this version of the metric would facilitate work on the Weyl metric in respect of Petrov Classification and singularities.

We consider the metric,

$$ds^2 = e^{2U}dt^2 - e^{2\upsilon+2\lambda-2U}(dx^{1^2} + dx^{2^2}) - e^{2\mu-2U}dx^{3^2}$$

$U = U(x^1, x^2)$, $\upsilon = \upsilon(x^1, x^2)$, $\lambda = \lambda(x^1, x^2)$, $\mu = \mu(x^1, x^2)$

where

$$\mu_{,11} + \mu_{,1}^2 - \mu_{,1}\upsilon_{,1} + \mu_{,2}\upsilon_{,2} = 0 \qquad \mu_{,22} + \mu_{,2}^2 + \mu_{,1}\upsilon_{,1} - \mu_{,2}\upsilon_{,2} = 0$$

$$\mu_{,12} + \mu_{,1}\mu_{,2} - \mu_{,1}\upsilon_{,1} - \mu_{,2}\upsilon_{,2} = 0 \qquad \upsilon_{,11} + \upsilon_{,22} = 0$$

are satisfied to make the spatial part flat with $U = 0$, $\lambda = 0$.

Let $X = \upsilon + \lambda - U$, $Z = \mu - U$ and $f = \mu - \upsilon - \lambda$

Then the metric takes the form

$$ds^2 = e^{2U}dt^2 - e^{2X}(dx^{1^2} + dx^{2^2}) - e^{2Z}dx^{3^2}$$

and the non-vanishing components of the Riemann Christoffel Tensor with respect to the above metric are,

$$R_{0101} = e^{2U}(U_{,11} + U_{,1}^2 - X_{,1}U_{,1} + X_{,2}U_{,2})$$

$$R_{0202} = e^{2U}(U_{,22} + U_{,2}^2 + X_{,1}U_{,1} - X_{,2}U_{,2})$$

$$R_{0102} = e^{2U}(U_{,12} + U_{,1}U_{,2} - X_{,1}U_{,2} - X_{,2}U_{,1})$$

$$R_{0303} = e^{2(U+f)}(Z_{,1}U_{,1} + Z_{,2}U_{,2})$$

$$R_{1212} = -e^{2X}(X_{,11} + X_{,22})$$

$$R_{1313} = -e^{2Z}(Z_{,11} - Z_{,1}^2 + Z_{,1}X_{,1} - Z_{,2}X_{,2} + 2Z_{,1}f_{,1})$$

$$R_{1323} = -e^{2Z}(Z_{,12} + Z_{,1}Z_{,2} + Z_{,1}X_{,2} - Z_{,2}X_{,1} + 2Z_{,1}f_{,2})$$

$$R_{2323} = -e^{2Z}(Z_{,22} - Z_{,2}^2 + Z_{,1}X_{,1} + Z_{,2}X_{,2} + 2Z_{,2}f_{,2})$$

The physical components of the Riemann Tensor with respect to the tetrad

$$T^{\mu}_{(0)} = \left(0, 0, 0, e^{-U}\right) \qquad T^{\mu}_{(1)} = \left(e^{U-\upsilon-\lambda}, 0, 0, 0\right)$$

$$T^{\mu}_{(2)} = \left(0, e^{U-\lambda-\upsilon}, 0, 0\right) \qquad T^{\mu}_{(3)} = \left(0, 0, e^{U-\mu}, 0\right)$$

$$R_{(11)} = R_{(0101)} = R_{0101}T^{0}_{(0)}\ T^{1}_{(1)}\ T^{0}_{(0)}\ T^{1}_{(1)} = e^{-2U}\ e^{2U-2\upsilon-2\lambda}\ R_{0101}$$

$$= e^{-2(\lambda+\upsilon)}R_{0101} = e^{2(U-\lambda-\upsilon)}\left(U_{,11} + U_{,1}^2 - X_{,1}U_{,1}\right) \quad \text{assuming } U_{,2} = 0$$

$$R_{(22)} = R_{(0202)} = e^{2(U-\lambda-\upsilon)}(X_{,1}U_{,1})$$

$$R_{(12)} = R_{(0102)} = e^{2(U-\lambda-\upsilon)}(-X_{,2}U_{,1})$$

$$R_{(33)} = R_{(0303)} = e^{2(U-\upsilon-\lambda)}(Z_{,1}U_{,1})$$

$$R_{(44)} = R_{(2323)} = e^{2(U-\upsilon-\lambda)}(-U_{,1}^2 + U_{1}(\lambda_{1} + \upsilon_{1} + \mu_{1}) - \mu_{1}\lambda_{1} + \mu_{2}\lambda_{2})$$

$$R_{(45)} = R_{(1323)} = e^{2(U-\upsilon-\lambda)}(U_{1}(\lambda_{2} + \upsilon_{2}) - \mu_{1}\lambda_{2} - \mu_{2}\lambda_{1})$$

$$R_{(55)} = R_{(1313)} = e^{2(U-\upsilon-\lambda)}(-U_{,11} + X_{1}U_{1} - \mu_{1}\lambda_{1} + \mu_{2}\lambda_{2})$$

$$R_{(66)} = R_{(1212)} = e^{2(U-\upsilon-\lambda)}(X_{,11} + X_{,22})$$

Now, we obtain the eigen values $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of the matrix

$$\begin{pmatrix} R_{(11)} & R_{(12)} & 0 \\ R_{(21)} & R_{(22)} & 0 \\ 0 & 0 & R_{(33)} \end{pmatrix}$$

$$\varepsilon_3 = R_{(33)} \quad , \quad \varepsilon_1 , \varepsilon_2 = \left\{ \left(R_{(11)} + R_{(22)} \right) \pm \sqrt{\Delta} \right\} \Big/ 2$$

where $$\Delta = \left(R_{(11)} + R_{(22)} \right)^2 - 4 \left(R_{(11)} R_{(22)} - R_{(12)}{}^2 \right)$$

These would be useful for the Petrov classification of metrics of the above type.