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Fourier method for one dimensional parabolic inverse problem with Dirichlet boundary conditions

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Abstract

The finite difference method, spectral method, and double shifted Lagrange's polynomials have been discussed for the one-dimensional inverse problem of the heat equation with control parameters and the source term in literature. Here, we present, Fourier method for the one-dimensional parabolic inverse problem with Dirichlet boundary conditions. In this study, after analyzing the control parameters, the initial condition and the source term are used to track a temperature distribution at a point in the interval. We validated that desired temperature distribution and measured temperature distribution (or the point evaluation) at an internal point overlap each other for the derived values of control parameters (source term and initial distribution) using the Fourier method. Moreover, we validated the temperature distribution at a point in the domain and tracked the desired harmonic and linear temperature distributions using numerical simulations. Finally, we simplified the above numerical simulations using the COMSOL software and illustrated some figures to the given point.

Keywords

Dirichlet boundary conditions, Fourier method, One dimensional heat equation control, Tracking problem, Point evaluation

Introduction

Heat conduction problems have achieved substantial popularity in science and industrial fields over the last five decades. Industry-related heat conduction research has great significance and is mostly regarded as the inverse problem of the heat conduction equation. Such as, by determining boundary, initial, or the internal data of the medium of heat transferred, a known temperature at the internal or the boundary point of the domain can be controlled. Usually, solving the inverse problem of the heat conduction equation is ill-conditioned and the small perturbation of data will lead to a problem with a huge error.

The inverse problem in (Grysa, 1980) discusses the dependence of the boundary conditions of different types on the prescribed temperature state. Moreover, an example of (Grysa, 1980) illustrates that how to make use of given temperature distribution on the surface to determine the thermal conditions of the heat medium near the sphere surface. (Dehghan, 2003) discusses a numerical approach for the one-dimensional parabolic inverse problem with control parameters. It presents several finite difference formulas for the inverse problem of finding a source parameter in the diffusion equation. Though the most usual way of generating a finite difference scheme is the use of Taylor's series, the method developed here is based on the modified equivalent partial differential equations. A direct computation technique for the inverse problem of finding a source term control parameter, using the spectral method can be found in (M Dehghan, 2006) Moreover there the function u(x,t) of two independent variables $0 \leq x \leq l$ and $0 \leq t \leq T$ has been expanded in terms of double shifted Lagrange's Polynomials. In (Ivanchov, Inverse Problem for General One Dimensional Parabolic Equation, 1998) the boundary integral method for solving one-dimensional homogeneous heat conduction equation has been discussed while (Ivanchov, One Dimensional Heat Conduction Equation for Inverse Problem, pp.60-66 2011) the existence and the uniqueness of the solutions of the one-dimensional inverse problem of the heat equation with time-dependent leading coefficient have been discussed. (Kanca, 2013) investigates the inverse problem of finding a time-dependent diffusion coefficient in a parabolic equation with the periodic boundary and integral overdetermination conditions. However, (Zhang & Guo, 2015)shows the numerical method is proposed to solve the inverse problem based on a Fourier expansion. Moreover, the articles (Hussein, Lesnic, & Ivanchov, 2014), (Negero & Tufa, 2014) and (Ivanchov, Inverse problems for Parabolic Equations, 2003) have been discussed on several sides of inverse problems for the parabolic equation. The novelty of our work is we use the Fourier method to construct the control parameters, initial condition, and the source term.

Consider the following uncontrolled heat equation with homogeneous Dirichlet boundary conditions in a finite-dimensional interval:

$$u_t(x,t) - u_{xx}(x,t) = 0 \qquad 0 \leq x \leq l \tag{1}$$
$$u(0,t) = u(l,t) = 0 \quad t \geq 0 \tag{2}$$
$$u(x,0) = \phi(x) \quad 0 \leq x \leq l \tag{3}$$

Find control parameters, initial temperature $g(x)$ and the heat source $f(x,t)$ such that point evaluation $u(x_0,t)$ tracks the desired signal $F(t)\epsilon\mathbb{C}[0,T]$ satisfying the system:

$$u_t(x,t) - \alpha u_{xx}(x,t) = f(x,t) \quad 0 < x < l\,, t > 0 \tag{4}$$
$$u(0,t) = u(l,t) = 0 \quad t \geq 0 \tag{5}$$
$$u(x,0) = g(x) \qquad 0 \leq x \leq l \tag{6}$$

Hence F(t) is a known function. In this research, we construct the control parameters, the initial condition, and source term so that the point evaluation at an internal point in the domain will track a known function. Finally, we validate our findings using the finite difference solver COMSOL as it provides better solution accuracy, consistency and easy meshing more efficiently and effectively.

Methodology

Consider the Fourier series representations of u(x,t) , f(x,t) and g(x) as follows.

Since u(x,t) which solves Equation(4) depends on both time and space, it can be represented as a Fourier sine series, $u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{l}\right)$ (7)

Notice that the u satisfy the boundary conditions in Eq.(5). Then the coefficient $a_n(t)$ is given by $a_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin\left(\frac{n\pi x}{l}\right)dx.$

We represented the right-hand side f(x,t) and the initial function g(x) in the same way as,
$f(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin\left(\frac{n\pi x}{l}\right)$; $c_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\left(\frac{n\pi x}{l}\right)dx.$ (8)

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right). \tag{9}$$

We setup the IVP for $a_n(t)$:

First, we can differentiate Eq.(7) with respect to t and x. Then we can derive the following equations to u_t, u_x, u_{xx} .

$$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t) \sin\left(\frac{n\pi x}{l}\right). \tag{10}$$

$$u_x = a_n(t) \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi}{l}\right).$$

$$u_{xx} = -a_n(t) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi}{l}\right)^2. \tag{11}$$

Plugging Equations (11), (10), (9), and (8) in Partial Differential Eq. (4), we can derive

$$\sum_{n=1}^{\infty} a_n'(t) \sin\left(\frac{n\pi x}{l}\right) + a_n(t) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left(\frac{n\pi}{l}\right)^2 = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{l}\right). \tag{12}$$

The above equation implies, $c_n(t) = a_n'(t) + a_n(t)\left(\frac{n\pi}{l}\right)^2$, $n = 1,2,3 \ldots\ldots$

According to the initial condition in Eq.(6), $u(x,0) = g(x)$. Thus we can derive the following,

$$\sum_{n=1}^{\infty} a_n(0) \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

$$b_n = a_n(0) \tag{13}$$

Then we solve the IVP Equation(13) and find $c_n(t)$ and b_n. Let $F(t)\epsilon\mathbb{C}(\mathbb{R},\mathbb{R})$ for all $t > 0$ be the reference signal to be tracked at the point $x = x_0$.Where $0 < x_0 < l.$ Next, we derive Equations for $c_n(t)$ and b_n in terms of $F(t)$. Let F(t) be the temperature distribution (reference signal) to be tracked at $x = x_0$.Then,

$$a_n(t) = \frac{2}{l}\int_0^l u(x_0,t) \sin\left(\frac{n\pi x_0}{l}\right) dx \quad ; \quad a_n(t) = \frac{2}{l}\int_0^l F(t) \sin\left(\frac{n\pi x_0}{l}\right) dx$$

Finally, we have $a_n(t) = 2\sin\left(\frac{n\pi x_0}{l}\right)F(t)$ (14)

Differentiating the above $a_n(t)$ w.r.t. t and plugging $a_n(t)$ and $a_n'(t)$ in IVP in Eq.(13)

$$c_n(t) = 2\sin\left(\frac{n\pi x_0}{l}\right)F'(t) + 2\left(\frac{n\pi}{l}\right)^2 \sin\left(\frac{n\pi x_0}{l}\right)F(t). \tag{15}$$

$$b_n = a_n(0) = 2\sin\left(\frac{n\pi x_0}{l}\right)F(0),\ n = 1,2,3 \ldots\ldots \tag{16}$$

Next, we prove the existence of one value for n. The orthogonal decomposition of f(x,t) can be written as $f(x,t) = c_1\varphi_1 + c_2\varphi_2 + \cdots$

where, $c_m = \frac{<\varphi_m,f>}{<\varphi_m,\varphi_m>}$; $\varphi_m = \sin\left(\frac{m\pi x}{l}\right)$.

Hence $f(x,t) = \sum_{n=1}^{\infty} c_n\varphi_n = \sum_{n=1}^{\infty} \frac{<\varphi_n,f>}{<\varphi_n,\varphi_n>}\varphi_n.$

Since $< \varphi_n, \varphi_m > = \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = 0 \quad \forall x \ \ n \neq m$

We have, $\quad < c_1\varphi_1 - f, \varphi_1 > = 0 \, , \ \forall x$

Finally, the above equation implies $f(x,t) = c_1(t)\sin\left(\frac{\pi x}{l}\right)$.

Example 1: Tracking F(t)=sin(t) at $x = x_0$.

Using the Eq.(15) Eq.(16) for b_n and c_n in the previous section,

$$c_1(t) = 2\sin\left(\frac{\pi x_0}{l}\right)\cos(t) + 2\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)\sin(t). \tag{17}$$

$$b_1 = 0 \tag{18}$$

$$f(x,t) = \sum_{n=1}^{\infty}\left[2\sin\left(\frac{\pi x_0}{l}\right)\cos(t) + 2\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)\sin(t)\right]\sin\left(\frac{\pi x}{l}\right). \tag{19}$$

$$g(x) = 0. \tag{20}$$

Example 2: Tracking F(t)=cos(t) at $x = x_0$.

Substituting Eq.(15),Eq.(16) we can obtain c_n and b_n for this case,

$$c_1(t) = -2\sin\left(\frac{\pi x_0}{l}\right)\sin(t) + 2\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)\cos(t). \tag{21}$$

$$b_1 = 2\sin\left(\frac{\pi x_0}{l}\right). \tag{22}$$

$$f(x,t) = \sum_{n=1}^{\infty}[-2\sin\left(\frac{\pi x_0}{l}\right)\sin(t) + 2\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)\cos\,(t)]\sin\left(\frac{\pi x}{l}\right). \tag{23}$$

$$g(x) = \sum_{n=1}^{\infty} 2\sin\left(\frac{\pi x_0}{l}\right)\sin\left(\frac{\pi x}{l}\right). \tag{24}$$

Example 3: Tracking F(t)=4t at $x = x_0$,

Then using the derived equations for b_n and c_n,

$$c_1(t) = 2\sin\left(\frac{\pi x_0}{l}\right)4 + 2\left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)4t. \tag{25}$$

$$b_1 = 0 \tag{26}$$

$$f(x,t) = \sum_{n=1}^{\infty} 8\left[\sin\left(\frac{\pi x_0}{l}\right) + \left(\frac{\pi}{l}\right)^2\sin\left(\frac{\pi x_0}{l}\right)t\right]\sin\left(\frac{\pi x}{l}\right). \tag{27}$$

$$g(x) = 0. \tag{28}$$

Results and Discussion

We set F(t)=sin(t) and solve

$$u_t(x,t) - u_{xx}(x,t) = 2\sin\left(\frac{\pi x_0}{l}\right)\sin\left(\frac{\pi x}{l}\right)[\cos(t) + \sin\,(t)\left(\frac{\pi}{l}\right)^2].$$
$$u(x,0) = 0. \qquad\qquad u(0,t) = u(l,t) = 0$$

On $x\epsilon[0,1]$ in COMSOL.Then we tracked the temperature distribution at the point x=0.75. The measured output is the point evaluation at x=0.75. The figure1 illustrates the desired temperature distribution and the measured output at x=0.75 in one figure. The figure1 shows that the desired temperature and the measured output overlap each other for the derived control parameters.

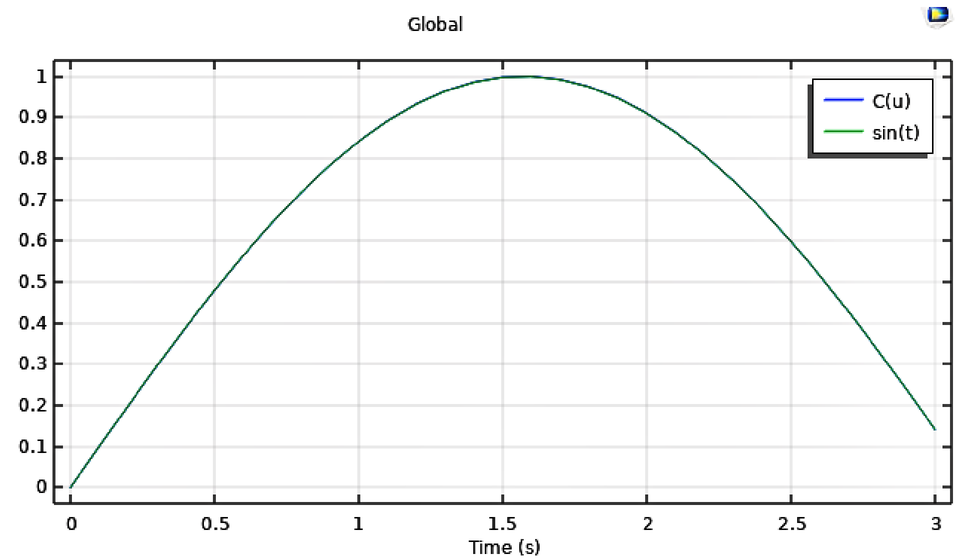

Figure 3. *The graph of sin(t) and measured output at x=0.75.*

Next, we consider F(t)=cos(t) and solve the following system.

$$u_t(x,t) - u_{xx}(x,t) = 2\sin\left(\frac{\pi x_0}{l}\right)\sin\left(\frac{\pi x}{l}\right)\left[\cos(t)\left(\frac{\pi}{l}\right)^2 - \sin(t)\right].$$

$$u(x,0) = 2\sin\left(\frac{\pi x_0}{l}\right)\sin\left(\frac{\pi x}{l}\right). \qquad u(0,t) = u(l,t) = 0.$$

On $x\epsilon[0,1]$ in COMSOL.Then we tracked measured output(point evaluation) at x=0.75. The figure 2 shows desired temperature distribution and the measured output at x=0.75 in one figure. The desired temperature and the measured output overlap each other for the derived control parameters as shown in figure2.

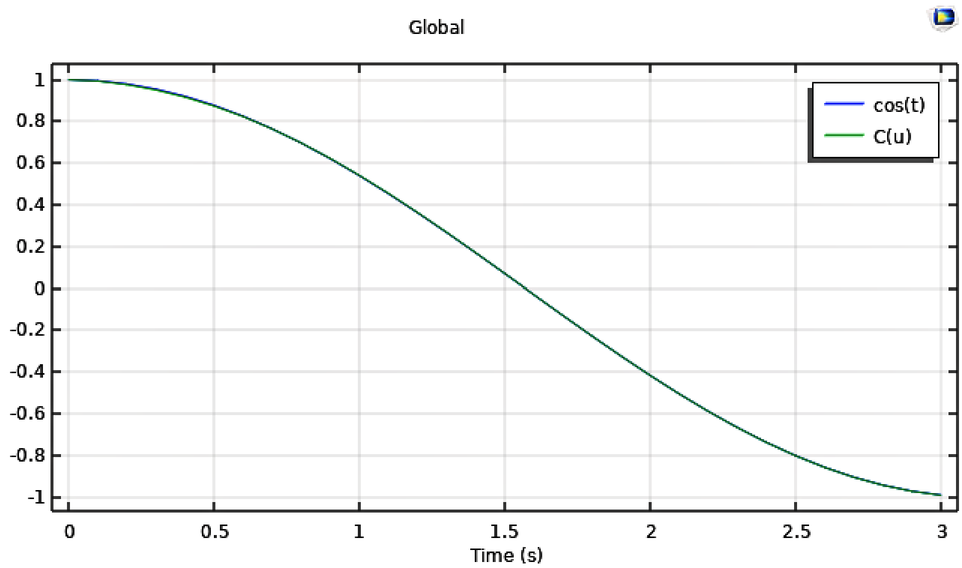

Figure 4. *The graph of cos(t) and the measured output at x=0.75.*

Considering F(t)=4t we solve,

$$u_t(x,t) - u_{xx}(x,t) = 8\sin\left(\frac{\pi x_0}{l}\right)\sin\left(\frac{\pi x}{l}\right)\left[1 + t\left(\frac{\pi}{l}\right)^2\right].$$
$$u(x,0) = 0. \qquad u(0,t) = u(l,t) = 0.$$

On $x\epsilon[0,1]$ in COMSOL. The measured output is the point evaluation at x=0.25. The figure3 illustrates desired temperature distribution and the measured output at x=0.25 in one figure. The figure3 clearly shows that the desired temperature and the measured output overlap each other for the derived control parameters.

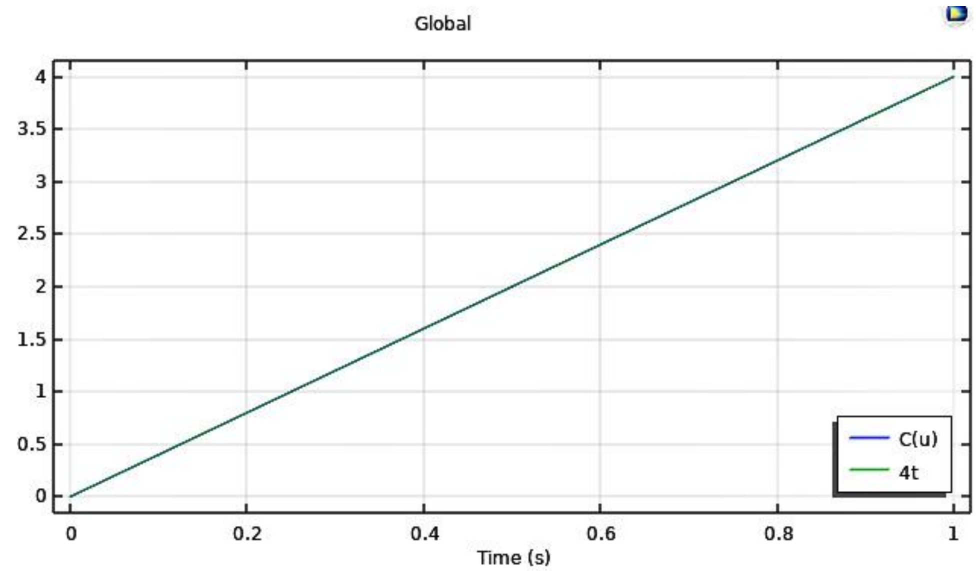

Figure 5. *The graph of 4t and the measured output at x=0.25.*

Conclusion

In this research work, we have considered the Fourier method for solving the one-dimensional parabolic inverse problem with Dirichlet boundary conditions where we first derived an ordinary differential equation for the Fourier coefficients. Then we expressed unknowns in our research problem, source term, and initial data in terms of those Fourier coefficients. Finally, we solved the heat equation on the one-dimension domain with the derived source term and the initial data. Furthermore, we validated that temperature distribution at internal points on the domain (x=0.25,x=0.75) tracked the harmonic and linear temperature distributions, using numerical simulations in COMSOL with the derived control parameters (source term and initial condition).

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