The Equality of Schrödinger's Theory and Heisenberg's S-matrix Theory

H. I. R. U. Silva, Dr. R.A.D. Piyadasa ,
Department of Mathematics,University of Kelaniya, Kelaniya,

ABSTRACT

It is well known that the Schrödinger's equation can be solved in few cases of physical importance [1] . Nevertheless, S-matrix theory can be used in general to describe physically important variables such as differential cross section, total cross section, etc….[2]. Since there's no any justification of theoretical work to the best of our knowledge to verify that the Schrödinger's theory and Heisenberg's S-matrix theory are equivalent in case of important interacting potentials for which the Schrödinger's equation can be solved analytically, we have used Heisenberg's S-matrix theory and Schrödinger's wave mechanics to justify that the two theories give exactly the same eigenvalues in cases which we have examined.

To obtain them, we were able to find the discrete energy eigenvalues in closed form in Heisenberg's theory without graphical methods.

In the case of Square-well potential, where $r < a$ and $V = -v_0$, partial wave $U_l(r)$ of angular momentum l, incident wave number k satisfies the Schrödinger's equation

$$\frac{d^2U_l(r)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right]U_l(r) = 0$$

(1)

where $\frac{2m}{\hbar^2}(\mathrm{E} + v_0) = k^2$ and we obtain

$$\xi^2 + \eta^2 = \gamma^2 \text{ and } \eta = \xi \tan \xi$$

(2)

Again biginning from equation (1) and the definition of S-matrix element one obtain, when $r = a$,

$$S_l(k) = \frac{U_l(a)H_l^{/(-)}(ka) - U_l^{/}(a)H_l^{(-)}(ka)}{U_l(a)H_l^{/(+)}(ka) - U_l^{/}(a)H_l^{(+)}(ka)}$$

(3)

When $l = 0$, $S_l(k)$ becomes zero where

$$k^2 = -\frac{2m}{\hbar^2}\left(v_0 + \frac{\hbar^2 k^2}{2m}\right)\tan^2\left\{\frac{a}{\hbar}\sqrt{2m\left(v_0 + \frac{\hbar^2 k^2}{2m}\right)}\right\}$$

(4)

which gives bound states.

Since $E = -\frac{\hbar^2 k^2}{2m}$ and $\frac{2ma^2}{\hbar^2}(v_0 - E) = y^2$

(5)

we obtain

$$y^2 + y^2 \tan^2 y - \frac{2mav_0 a^2}{\hbar^2} = 0$$

(6)

This implies that the Heisenberg's equation can be solved in closed form for energy eigenvalues without depending on numerical work.

Also, we have justified that the equations (2) and (4) are same as the equation (6).

By using Parabolic co-ordinates to solve the Schrödinger's equation for the Hydrogen Atom and using Hyper Geometric Confluent functions, we have expressed the S-matrix element in terms of Gamma functions as

$$S_l^n(k) = \frac{\Gamma(l+1+in)}{\Gamma(l+1-in)} \tag{7}$$

Then it was apparent that the S-matrix element contains infinite number of poles and zeros.

Considering the relevant simple pole, we can derive an equation for the energy eigenvalues.

This leads to a general formula for energy values as

$$E_n = -\frac{\mu e^4}{2\hbar^2 n^2} \text{ where } n = N + l + 1 \tag{8}$$

This shows that it is same as the equation we obtain in Schrödinger's theory.

References

(1) Schiff L. I. 1949, "Quantum Mechanics",1st edition, pp35-119.

(2) Schwabl F. translated by Kate R. 1998, "Quantum Mechanics", 3rd edition, pp320-333.