A simple analytical proof of Fermat's last theorem for $n = 7$

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(Dedicated to late Professor S.B.P.Wickramasuriya)

ABSTRACT

It is well known that proof of Fermat's last theorem for any odd prime is difficult and first proof for $n = 7$ was given by Lame [1] ,and Kumar also has given a proof for a special class of primes (Regular primes)which includes the case $n = 7$ .However, these proofs are lengthy and difficult and may not easily be extended for all odd primes. The prime $n = 7$ differs from $n = 5$ since $2.7.+1 = 15$ is not a prime, whereas $2.5+1 = 11$ is a prime. Then it follows from the famous theorem of Germain Sophie that the corresponding Fermat's equation $z^7 = x^7 + y^7, (x, y) = 1$ may have two classes of integer solutions, $xyz \equiv 0(\text{mod}7)$ and $xyz \neq 0(\text{mod}7)$ if we assume that the Fermat equation has non-trivial integer solutions for x, y, z . This fact is proved using the simple argument [3] of Oosterhuis. The main objective of this paper is to give a simple analytical proof for the Fermat's last theorem $n = 7$ using general respective parametric solutions corresponding two classes of solutions of the Fermat equation ,which has already been extended for all odd primes.

Parametric solutions of the Fermat equation

Let us first show that the Fermat equation

$$z^7 = y^7 + x^7, (x, y) = 1 \quad (1)$$

may have two classes of solution for x, y, z if we assume that it has non-trivial solutions. If $xy \neq 0(\text{mod}7)$, then we can write $x = 7k \pm 1, 7l \pm 2, 7q \pm 3$, $y = 7r \pm 1, 7s \pm 2, 7t \pm 3$ and therefore $x^7 + y^7 \equiv (0, \pm 2, \pm 19, \pm 18, \pm 16)(\text{mod } 7^2)$ .But $z^7 \equiv (1, -1, 19, -19, 16, -16)(\text{mod } 7^2)$ when $z \neq 0(\text{mod}7)$ and we conclude that there may be class of non-trivial solutions for x, y, z, one such that $xyz \neq 0(\text{mod}7)$ and the other such that $xyz \equiv 0(\text{mod}7)$ These solutions are called generally Class.1 and Class.2 solutions respectively.

Structure of Fermat triples

Class.1

The structure of Fermat triples has been obtained by many authors[1].The following form of the Fermat triples is sufficient for our purpose, and in the following, the structure of Fermat triples is obtained for the general prime exponent p using Fermat's little theorem.

It is not a difficult at all to obtain the relations $z - y = h^p$, $z - x = u^p$, $x + y = g^p$, where h, u, g are factors of x, y, z, respectively[1] in case of Class.1 solutions. Since

$$z^p - z = y^p - y + x^p - x + (x + y - z) \quad (2)$$

we obtain $x + y - z \equiv 0(\text{mod} p)$ using Fermat's little theorem. If we write $z = gs$,

$$x + y - z = g^p - gs = g[g^{p-1} - 1 - (s - 1)] \equiv 0(\text{mod } p) \quad (3)$$

and we deduce at once $s \equiv 1(\text{mod} p)$ .Therefore $z = g(pk + 1)$,where $s = pk + 1$ and in the same manner, we obtain $x = h(pl + 1)$, $y = u(pq + 1)$ .Now,

$$2(x + y - z) = g^p - u^p - h^p = (z - gpk)^p - (y - upq)^p - (x - hpl)^p \equiv 0(\text{mod } p^2) \quad (4)$$

which follows at once since p is an odd prime and $z^p - y^p - x^p = 0$. Now, we use the relation

$$x-(z-y)=y-(z-x)=(x+y)-z \quad (5)$$

to deduce that $x+y-z \equiv 0(\bmod\ guhp^2)$ .In particular $x+y-z=x-h^p$. In case of $p=7$, we use Werbrusow equation [1]

$$(x+y-z)^7 = 7(x+y)(z-x)(z-y)d^7 \quad (6)$$

where

$$d^7 = \frac{6!}{2^5}[(x+y)^4+(z-x)^4+(z-y)^4+(z-x)^2((z-y)^2+(x+y)^2)+(z-y)^2(x+y))^2]$$

to deduce $x+y-z=ughd7^m$,and it is easy to understand that d is odd., and as in the case of $n=5$, d is co-prime to ugh .Therefore , it follows from (4) that

$$g^7 - 2ughd7^m - u^7 - h^7 = 0 \quad (7)$$

where $m \geq 2$ and the positive integers $u,.g,h,d$ are co-prime to one another

Class .2

Assume that $y \equiv 0(\bmod p)$ and all x,y,z are positive. Then, from the Fermat equation

$$x^p + y^p = z^p, (x,y)=1 \quad (8)$$

parametric solution corresponding to (5) and $y \equiv 0(\bmod p)$ can be written[4]as

$$g^7 - p^{7m-1}u^7 - h^7 - 2.7^m ughd = 0 \quad (9)$$

and this is the necessary condition satisfied by the factors of x,y,z and the odd number d co-prime to x,y,z.

Proof of the theorem

Let us first consider Class.1 solutions. In case of Class.1, the necessary condition

$$g^7 - 2ughd7^m - (h^7+u^7) = 0 \quad (10)$$

is a polynomial equation of seventh degree in g and we fix the factors $m.,u$ of y in the Fermat equation and look for the integral solution of (10) corresponding to g for different h, a factor of x. If we fix y in the Fermat equation, z is given by $z=y+h^7$ and x is given by $x=z-7^{7m-1}u^7$.Then d is determined by the Darbruosow equation. In (10), u,h,g,d are co-prime numbers and it must have an integer root when we fix the numbers except g. Integer roots of the equation are the integral factors of h^7+u^7. Now, we show that g can not be $h+u$. Using the identity [1],

$$h^7+u^7 \equiv (h+u)^7 - 7uh(h+u)(h^2+uh+u^2)^2 \quad (11)$$

we write (8) in the form

$$g^7-(h+u)^7 - 2ughd7^m + 7uh(h+u)(h^2+uh+u^2)^2 = 0 \quad (12)$$

If $g=h+u$, then we get from (10) that

$$-2.d.7^m + 7(h^2+uh+u^2)^2 = 0 \quad (13)$$

It is obvious that $(h^2+uh+u^2)^2$ is odd since u,h are either of opposite parity or both are odd. Therefore (10) is never satisfied and hence $g \neq h+u$. It is clear that, since g is co-prime to 7 , $h+u$ is also co-prime to 7. Hence, it follows from (10) that $g-(h+u) \equiv 0(\bmod 7)$ since it can be written as

$$g^7 - g - 2ughd7^m - (h^7 - h + u^7 - u) + g - (h+u) = 0 \quad (14)$$

and due to Fermat's little theorem. Therefore let $g=h+u+7^k t$ where $k \geq 1$ and $t \neq 0$. Let us now write (10) in the form

$$[(h+u+7^j t)^6 - 2uhd7^m](h+u+7^j t) = h^7+u^7 \quad (15)$$

Since $h+u+7^j t$ is a factor of h^7+u^7, it should w vanish when $h=-(u+7^j t)$ vanish. Hence, we must have

$$(u+7^j t)^7 = u^7 \quad (16)$$

for non-zero t, which is never satisfied. Hence, (8) can not be satisfied by integral parameters and hence the Fermat equation (1) has no Class.1 solutions.
Now, consider the Class.2 solutions. In this case, we must have

$$g^7 - h^7 - 2.7^m ughd - 7^{7m-1} u^7 = 0 \quad (17)$$

As in case of $n=5$[5] we must have $g = h + 7^{m-1} j$, where $(7, j) = 1$. Hence, we get

$$[(h+7^{m-1} j)^6 - 2uhd7^m](h+7^{m-1} j) = h^7 + 7^{7m-1} u^7 \quad (18)$$

from which we get

$$-7^{7m-7} j^7 + 7^{7m-1} u^7 = 0 \quad (19)$$

It is obvious that (19) is never satisfied since $(7, j) = 1$. Therefore we conclude that the Fermat equation has no Class.2 solutions as well. This completes the proof of Fermat's last theorem for $n=7$.

References

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(5) R.A.D.Piyadasa, Simple analytical proofs of Fermat's last theorem for $n=5$ and for many odd primes,CMNSEM,Vol.1 No.5,July 2010, p.127-131