Interior solution to a celestial object

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ABSTRACT

With a suitable choice of coordinates the internal space time due to a sphere of fluid with radius r_0 , can be expressed as, $ds^2 = e^{\upsilon} c^2 dt^2 - \left[e^{\lambda} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \ d\varphi^2\right)\right]$; where $\upsilon(r)$ and $\lambda(r)$ are functions of the radial variable r, which must be determined from the three field equations given below.

$$\rho = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2} \tag{1}$$

$$\frac{CP}{c^2}\ \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2}\ \frac{\upsilon'}{r}\right) \tag{2}$$

$$\frac{CP}{c^2} = e^{-\lambda}\left[\frac{\upsilon'\lambda'}{4}\ \frac{\upsilon'^2}{4}\ \frac{\upsilon''}{2}\quad \frac{(\upsilon' - \ \lambda'\ )}{2\,r}\right] \tag{3}$$

Where $C = \frac{-8\pi G}{c^2}$, G is the gravitational constant and c is the velocity of light in a vacuum.

Equation (1), (2) and (3) has been solved for constant density ρ_0 with the condition P = 0 at the boundary [(1)]. In this presentation we solve the field equations to obtain an expression for the interior solution of an object when the fluid sphere has a constant density ρ_0 and a variable pressure P(r), without taking P = 0 at the boundary.

By solving the equation (1), e^{λ} can be expressed as follows.

$$e^{\lambda} \frac{1}{1+\frac{C\,\rho_0 r^2}{3}} \tag{4}$$

Also e^{λ} can be expressed by solving the second and third equations as,

$$e^{\lambda} = \frac{\left(\frac{5P}{c^2}+\rho_0-\frac{2}{rc^2}\frac{d(Pr^2)}{dr}\right)}{\left(\rho_0+\frac{P}{c^2}\right)\left(1-\frac{C\,P\,r^2}{c^2}\right)} \tag{5}$$

By equating (1) and (2), the pressure inside the spherical rigid object can be expressed as,

$$P(r) = \frac{\rho_0 c^2}{3}\left(\frac{3\,e^{k\left(1+\frac{\rho_0 C r^2}{3}\right)^{-\frac{1}{2}}} - 1}{1-\ e^{k\left(1+\frac{\rho_0 C r^2}{3}\right)^{-\frac{1}{2}}}}\right)$$

where k =

$$\ln\left(\frac{\mathrm{P}_0+\frac{\rho_0 c^2}{3}}{\mathrm{P}_0+\rho_0 c^2}\right).\left(1+\frac{\rho_0 C {r_0}^2}{3}\right)^{\frac{1}{2}} \tag{6}$$

By equating the coefficient of dr^2 expressed in the above interior solution to the Schwarzschild exterior solution [1] at the boundary the radius of the fluid sphere r_0 can be expressed as,

$$r_0 = \left(\frac{3\,m_0 c^2}{4\,\pi\,\rho_0 G}\right)^{\frac{1}{3}} \tag{7}$$

By solving the field equations (1), (2), (3) we obtain the following expression e^{v} in terms of P(r),

$$e^{\upsilon} = \left(1 - \frac{2m_0}{r_0}\right) + \frac{\left(2 + \frac{(P_0 + P)}{\rho_0 c^2}\right)\left(\frac{P_0 - P}{\rho_0 c^2}\right)}{\left(1 + \frac{P}{\rho_0 c^2}\right)^2 \left(1 - \frac{P_0}{\rho_0 c^2}\right)^2} \quad (8)$$

where P_0 is the fluid pressure at the boundary.

e^{υ} can be expressed as a function of r, by substituting for P in (8), using (6).

References:

1. Alder Ronald, Bazin Maurice, Schiffer Menahem; Introduction to General Relativity; McGraw-Hill Book Company. (1965)
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