3.8 Integer roots of a polynomial equation and Fermat's last theorem for n=3.5.

R.A.D.Piyadasa
Departmentof mathematics,Universiy of Kelaniya,Kelaniya,
Kelaniya,Sri Lanka.
E-mail: Piyadasa54@yahoo.com

ABSTRACT

Integer roots of a polynomial equation and Fermat's last theorem for n=3, 5.

R.A.D.Piyadasa, Department of mathematics, University of Kelaniya

(Dedicated to late professor S.B.P.Wickramasuriya)

We have found that a simple theorem[1] is capable discarding all integer solutions of a special class of prime degree polynomial equations. In this contribution a special form of a cubic and quintic is explored using this theorem and Fermat's last theorems for $n = 3,5$ are easily proved. It should be noted that Fermat's last theorem for $n = 3$ has been proved by Euler in 1770. But in his proof he did not fully establish the key lemma required in the proof. Paulo Rebenboin[2] claims that he has patched up Euler's proof and Edwards[3] has also given a proof of the Key lemma of Euler's proof using a Ring of complex numbers. However, in this author's point of view, none of these proofs is easy to understand. Proof of Fermat's last theorem for $n = 5$ has shared by two eminent mathematicians, Dirichlet and Legendre. Their proof, however, is difficult and lengthy. The main objective of this contribution is to provide simple proofs for the Fermat's last theorems for $n = 3,5$.

A special cubic related to Fermat's last theorem for n=3.

We have obtained [4]the parametric solution of the equation,

$$z^3 = y^3 + x^3,\ (x, y) = 1. \tag{1}$$

in the form

$$x = 3^m ugh + h^3 \tag{a}$$

$$y = 3^m ugh + 3^{3m-1} u^3 \tag{b}$$

$$z = 3^m ugh + 3^{3m-1} u^3 + h^3 \tag{c}$$

where we have assumed that y is divisible by $3.$ This assumption we have made is quite general since x, y, z in (1) can be replaced by $-x,-y,-z$ .Parameters in (a),(b) and (c) must satisfy the necessary condition that

$$g^3 - h^3 - 2.3^m ugh - 3^{3m-1} u^3 = 0 \tag{d}$$

In this equation $u, g, h, 3$ are co-prime to one another and the positive integer $m \geq 2$. One can prove Fermat's last theorem for $n = 3$ by showing that there is no integer root g satisfying (d) for fixed u, h, m.

1. Proof of Fermat's last theorem for n=3.

This equation is of the form

$$g^3 - 3uvg - v^3 - w^3 = 0 \qquad (2)$$

where $uv = 2uh3^{m-1}$, $v^3 + w^3 = 3^{3m-1}u^3 + h^3$ and its roots for g are given by[5] $v + w, v\varpi + w\varpi^2, v\varpi^2 + w\varpi$, ω being the cube root of unity .Discriminant Δ of the equation is $\Delta = -[3^{6m-2}u^6 - 14.3^{3m-3}u^3h^3 + h^6] = -[(3^{3m-1}u^3 - h^3)^2 + 4.3^{3m-3}u^3h^3]$,which is negative when $uh > 0$ or $uh < 0$.Hence, the equation has only one real root[5],namely , $v + w$.Due to a lemma proved in ref..1 the equation has an integer root of the form $3^{m-1}j + h$ unless $uh = 0$ .Now, we will show that $u = \pm 1, h = \pm 1$ are not accessible. Assume, for example $u = 1, h = 1$.Then the integer solution must have the form $3^{m-1}.2 + 1$ or $3^{m-1} + 2$. Then we must have $(3^{m-1}.2)^3 + 1^3 = 3^{3m-1} + 1$ or $(3^{m-1})^3 + 2^3 = 3^{3m-1} + 1$, which never holds. Therefore the equation has no integer roots in this case. It is easy to verify that the equation has no integer solutions for all values of $u = \pm 1, h = \pm 1$. Now assume that $h \neq \pm 1$.Then $g = 3^{m-1}j + h$ is the integer real root of the equation. In order for $g = 3^{m-1}j + h$ to be an integer root of the equation , $3^{m-1}j + h$ must be a factor of $3^{3m-1}u^3 + h^3$,which follows from (d). As a result of this, we can write

$$3^{3m-1}u^3 + h^3 = (3^{m-1}j + h)[(3^{m-1}j + h)^2 - 2.3^m uh] \qquad (3)$$

In other words, $-3^{m-1}j$ is a root of the polynomial equation $3^{3m-1}u^3 + h^3 = 0$ in h.From which, we obtain $3^{3m-1}u^3 - 3^{3m-3}j^3 = 0$ and hence $9u^3 = j^3$.This contradicts our assumption that j is an integer and therefore (d) has no integer roots. This completes the proof of Fermat's last theorem for $n = 3$.

2. Proof of Fermat's last theorem for n=5.

It can be shown ,as in the case of $n = 3$ that, in order to prove the Fermat's last theorem for $n = 5$, one has to prove that the equation

$$g^5 - h^5 - 2.5^m ugh - 5^{5m-1}u^5 = 0 \qquad (4)$$

has no integer roots for g , where $u, g, h, 5$ are co-prime to one another and the positive integer $m \geq 2$.Proof is exactly the same as in the case of $n = 3$ and we deduce that integer solutions of the equation is of the form $g = 5^{m-1}j + h$. Also, we must have

$$5^{5m-1}u^5 + h^5 = (5^{m-1}j + h)[(5^{m-1}j + h)^4 - 2.5^m uh] \qquad (5)$$

Therefore $h = -5^{m-1}j$ must satisfy the polynomial equation in h given by

$$5^{5m-1}u^5 + h^5 = 0 \qquad (6)$$

In other words , we must have $5^4u^5 = j^5$ and hence j can not be an integer. Therefore (4) has no integer solutions and our proof is complete. It should be noted that the solutions corresponding to $h = \pm 1, u = \pm 1$ can be discarded easily in this case as well.

References

(1) Piyadasa R.A.D. On the integral roots of special class of prime degree polynomial Equations,to be published, 2009,Annual research symposium, University of Kelaniya,Kelaniya.

(2) Paulo Ribenboim ,Fermat’s last theorem for amateurs, Spriger-Verlag-New-York,1999

(3) Harold M.Edwards,Fermat’s Last Theorem ,Springer-Verlag-New York,1977

(4) Piyadasa R.A.D.,(2007) .Analytical proof of Fermat’s last theorem for n=3 , Annual research symposium of faculty of graduate studies, University of Kelniya,Kelaniya. 127-128

(5) Archbold,J.W.1961 . Algebra ,London Sir Issac Pitmann & Sons LTD, pp.174