# 3.13 On the validity of a practical three – body model

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## ABSTRACT

It is well known among Physicists that the classical three – body problem is not solvable, whereas, the Quantum Mechanical three–body problem is solvable due to the famous Faddeev's work [1]. However, the problem in Faddeev's method is not a practical method since it is not directly applicable to the simplest three–body problem. In particular, the important Coulomb potential cannot be included in a mathematically rigorous manner.

Kushu group [2] developed a practical method based on [3] which has been remarkably well [4] in producing experimental results, and is now used all over the world, in case of elastic scattering of lights ions such as d, Li, etc. Which are easily breakable in scattering on composite nuclei. This method (CDCC) is simpler and it solves quantum mechanical Shrödinger equation corresponding to the three–body problem concerned. Very concise recap of this model is given in case of the three–body model $n - p - A$ ($d - A$) in the following.

The total wave function $\Psi$ of the 3–body n–p–A system associated with the model, Hamiltonian $H$ is expressed by

$$H = \sum_{JM} a_{JM}\ R^{-1}\ \psi_{JM}$$

in the usual notation.

Now $\psi_{JM}$ is expanded in the complete set of eigen functions of the deuteron sub Hamiltonians

$$H_{np} = K_r + V_{np}(r)$$

in the usual notations.

Here the ground–state wave function of $\phi_d(r)$ and continuum set of wave functions $\{\phi_l(k,r)\}$ play a vital role.

Now

$$\psi_{JM} = \chi_{JM}(P_0,R)\ \phi_d(r)\ Y_{JM}(\hat{R})\ Y_{00}(\hat{r}) \times i^J + \sum_{l=0}^{\infty}\sum_{L}\int_0^{\infty} \phi_l(k,r)\ \chi_{lL}^{J}\big(P(k),R\big)dr \times \left[Y_L(\hat{R})\ i^L Y_l(\hat{r})\ i^l\right]_{JM}$$

in the usual notation.

$E_t$, the total centre of mass energy, is given by

$$E_t = \frac{\hbar^2 P_0^2}{2\mu_d} + \varepsilon_0 = \frac{\hbar^2 P(k)^2}{2\mu_d} + \frac{\hbar^2 k^2}{m_N}$$

in the usual notation.

Some assumptions, further, are needed. One of which in the cut off of the continuum and consider the Riemann sum over $[0, k_1], [k, k_2], \ldots\ldots [k_i, k_{i+1}], \ldots\ldots [k_{N-1} - k_N = k_m]$.

Still further, the following assumption is needed to do computer calculations.

$$\int_{k_i}^{k_{i+1}} \phi_l(k,r)\ \chi^J_{lL}\big(P(k),R\big)dk = \sqrt{\Delta}\ \chi^J_{ilL}\Big(\hat{P}(k_i),R\Big)\hat{\phi}_{il}(r)$$

where $\hat{\phi}_{il}(r) = \int_{k_i}^{k_{i+1}} \phi_l(k,r)\ dk \big/ \sqrt{\Delta}$

This averaging procedure was drastically criticised by the experts [4] of Faddeev theory. The criticism was so drastic and that one had to answer at least on Physical grounds and which was done in [5]. The above criticism was fully answered, mainly on physical grounds, by the authors of [6] doing a the then gigantic numerical calculation. It has been now shown [7] also that CDCC method is the first order approximation to the Faddeev method. Then the question is why the first order method work so well. Answer to this question is mainly [8] and [9]. The main purpose of this paper is to justify, to a certain extent, CDCC, in a mathematical rigorous manner, by producing the correct form, which has been scrutinized by the authors, of the potential tails of CDCC and numerical support as in the following.

Continuum – Continuum coupling potential $V_{k,k}(R)$, in the usual notation, can be written as

$$V_{k,k'}(R) = \int_0^\infty U_0(k,r)\ V_0(R,r)\ U_0(k',r)\ dr$$

(1)

in the usual notation for the simplest case of CDCC, where

$$U_0(k,r) = r\ \phi_0(k,r)$$

(2)

Here $\phi_0(k,r)$ defines the deuteron S – state breakup wave function of linear momentum $k$. Now

$$U_0(k,r) = \sqrt{\frac{2}{\pi}}\ \sin\big(kr + \delta(k)\big)$$

(3)

Neglecting $\delta(k)$, the phase shift, for the sake of simplicity, one writes

$$V_{k,k'}(R) = \frac{1}{2}\int \big[\ \cos\big(k-k'\big)r - \cos\big(k+k'\big)r\ \big]\ V_\lambda(R,r)\,dr$$

(4)

$V_\lambda(R,r)$ here has the usual meaning.

In case of square well potential

$$V_{k,k'}(R) = \frac{V_{0,0}}{TR}\ \int_{-a}^{a} \frac{\sin\ 2k(R+r)\ \sin 2k'(R+r)\ (a^2 - r^2)}{\big(R+r\big)}\ dr$$

(5)

This can be readily simplified to

$$V_{k,k'}(R) = \frac{2V_{0,0}a^3}{3\pi R^2}\left[1 + \left(\frac{3}{16}\right)c^2\cos 4kR\cos 4ka \quad - \quad \left(\frac{3}{64}\right)c^3\cos 4kR\sin 4ka\right]$$

(6)
when $k = k'$, under the assumption $R >> a$. Here $c$ given by $cka = 1$.
If $ka >> 1$,

$$V_{k,k'}(R) = \frac{2V_{0,0}\, a^3}{3\pi R^2}$$

(A)
When $k = k'$

$$V_{k,k'}(R) = \frac{2\,V_{0,0}\; a^3}{\pi\; R^2}\left\{\left[\frac{\cos 2k'R\,\cos 2k'a}{(2k'a)^2} - \frac{\cos 2kR\,\cos 2ka}{(2ka)^2}\right] - \left[\frac{\cos 2k'R\,\sin 2k'a}{(2k'a)^3} - \frac{\cos 2kR\,\sin 2ka}{(2ka)^3}\right]\right\}$$

(B)
where $K' = k' - k$ and $K = k' + k$
(A) and (B) agrees with numerical calculations very nicely, which is depicted by the figures attached, in case of realistic potentials. In the figure 1, the diagonal potential (1 – 1) , (6 – 6) agree exactly the form, mathematically established, and figure 2 in case of non-diagonal potentials.

**References**


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