## 3.11 Convergence of multivariate isotonic regression

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**ABSTRACT**

Statistical inference in the presence of order restrictions is an important area of statistical analysis. Isotonic regression theory plays a key role in this field.

Let $K=\{1,\ldots,k\}$ be a finite set on which a partial order « is defined. A real vector $(\theta_1, \ldots,\theta_k)$ is said to be isotonic if $\mu, \upsilon \in K$, $\mu$ « $\upsilon$ imply $\theta_\mu \leq \theta_\upsilon$. Given real numbers $x_1,\ldots,x_k$ and positive numbers $w_1,\ldots,w_k$, a vector $\left(\widehat{\theta}_1,\ldots,\widehat{\theta}_k\right)$ is said to be the univariate isotonic regression of $x_1,\ldots,x_k$ with weights $w_1,\ldots,w_k$ if it is isotonic and minimizes

$$\sum_{\nu=1}^{k}(x_\nu - \theta_\nu)^2 w_\nu$$

under the restriction that $(\theta_1,\ldots,\theta_k)$ is isotonic. Isotonic regression is closely related to the maximum likelihood estimate of ordered parameters of univariate normal distribution and some other univariate distributions. Various algorithms are given in the literature for computing univariate isotonic regression.

Multivariate generalization of the isotonic regression and multivariate extensions of related theorems are given and proved by Sasabuchi, Inutsuka and Kulatunga (1983, 1992).

A $p \times k$ real matrix $\theta = (\theta_1,...,\theta_k)$ is said to be isotonic with respect to the partial order « , if $\mu, \upsilon \in K$, $\mu$ « $\upsilon$ imply $\theta_\mu \le \theta_\nu$, where $\theta_\mu \le \theta_\nu$ means all the elements of $\theta_\nu - \theta_\mu$ are nonnegative .

Given p-dimensional real vectors $x_1,\ldots,x_k$ and $p \times p$ positive-definite matrices $\Lambda_1,\ldots,\Lambda_k$, a $p \times k$ matrix $\left(\widehat{\theta}_1,\ldots,\widehat{\theta}_k\right)$ is said to be the multivariate, in fact *p*-variate, isotonic regression of $x_1,\ldots,x_k$ with weights $\Lambda_1^{-1},\ldots,\Lambda_k^{-1}$ if it is isotonic and satisfies

$$\min{}_\theta^* \sum_{\nu=1}^{k}(x_\nu - \theta_\nu)'\Lambda_\nu^{-1}(x_\nu - \theta_\nu) = \sum_{\nu=1}^{k}\left(x_\nu - \widehat{\theta}_\nu\right)'\Lambda_\nu^{-1}\left(x_\nu - \widehat{\theta}_\nu\right),$$

Where $\min_\theta^*(.)$ to denotes the minimum for all $\theta$ isotonic with respect to the partial order. An algorithm for the computation of multivariate isotonic regression is given in Sasabuchi *et al*. (1983, 1992). This algorithm involves iterative computation of univariate isotonic regression. The convergence of this algorithm is also studied there and it has been observed that the convergence follows only under certain conditions. (Corollary 4.1 of Sasabuchi *et al* (1992).)

However, the simulation study conducted in Sasabuchi, Miura and Oda (2003) under special cases, has shown that the condition given in Corollary 4.1 of Sasabuchi *et al*. (1992) is not necessary for the convergence of this algorithm. We have written a Fortran subroutine for the computation of multivariate isotonic regression and also noted that the algorithm converges in general. This motivates us to consider a proof for the convergence of this algorithm.

In this study we give a proof for the convergence of this algorithm in the bivariate case.

**References**

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