4.23 A metric which represents a sphere of constant uniform density comprising electrically counterpoised dust

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ABSTRACT

Following the authors who have worked on this problem such Bonnor et.al[1,2], Wickramasuriya[3] and we write the metric which represents a sphere of constant density $\rho = \frac{1}{4\pi}$, with suitable units, as

$$ds^2 = \frac{1}{(\theta(r))^2} c^2 dt^2 - (\theta(r))^2 \left(dr^2 + r^2 d\Omega^2\right) \qquad 0 \le r \le a$$

$$ds^2 = \frac{1}{\left(D + \frac{B}{R}\right)^2} c^2 dT^2 - \left(D + \frac{B}{R}\right)^2 \left(dR^2 + R^2 d\Omega^2\right) \qquad A < R$$

where $d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right)$, $\theta(r)$ is the Emden function satisfying the Emden equation[4] with $n = 3$. Since the metric has to be Lorentzian at infinity, we can take $D = 1$. However, there is an important difference between the above authors and us as they had taken the same coordinate r in both regions, and as a result $A = a$. In general these coordinates do not need to be the same. In this particular case the coefficients of $d\Omega^2$ are not of the same form in the above two metrics and that forces us to take two different coordinates r and R. In our approach $r=a$ in the matter-filled region corresponds to $R = A$ in the region without matter.

Applying the boundary conditions at $r = a$ or $R = A$, we have,

$$\frac{1}{\theta(a)} c\, dt = \frac{1}{\left(1 + \frac{B}{A}\right)} c\, dT$$

$$\Rightarrow \frac{dt}{dT} = \frac{\theta(a)}{\left(1 + \frac{B}{A}\right)} \qquad \text{(i)}$$

$$\frac{-2}{(\theta(a))^3} \theta'(a) c dt = \frac{-2}{\left(1 + \frac{B}{A}\right)^3} \left(-\frac{B}{A^2}\right) c dT$$

$$\Rightarrow \frac{dt}{dT} = \frac{-B(\theta(a))^3}{A^2 \theta'(a) \left(1 + \frac{B}{A}\right)^3} \qquad \text{(ii)}$$

$$\theta(a) dr = \left(1 + \frac{B}{A}\right) dR \qquad \Rightarrow \frac{dr}{dR} = \frac{\left(1 + \frac{B}{A}\right)}{\theta(a)} \qquad \text{(iii)}$$

$$\theta(a)a = \left(1+\frac{B}{A}\right)A \_\_\_\_\_\_\_\_\_\_\text{(iv)}$$

From (i) and (ii), we have $\dfrac{\theta(a)}{\left(1+\frac{B}{A}\right)} = \dfrac{-B(\theta(a))^3}{A^2\theta'(a)\left(1+\frac{B}{A}\right)^3}$

$$B = -\frac{A^2\left(1+\frac{B}{A}\right)^2\theta'(a)}{(\theta(a))^2} \_\_\_\_\_\_\_\_\_\_\text{(v)}$$

From (iv), $\dfrac{\left(1+\frac{B}{A}\right)}{\theta(a)} = \dfrac{a}{A}$ __________(vi)

Using equation (vi) in equation (v), we have $B = -A^2\left(\dfrac{a^2}{A^2}\right)\theta'(a) = -a^2\theta'(a)$.

Substituting the value of B in equation (iv), $\theta(a)a = \left(1+\dfrac{B}{A}\right)A = A + B$

$$= A - a^2\theta'(a)$$

$$\Rightarrow A = a\theta(a) + a^2\theta'(a)\ .$$

Then the metric becomes

$$ds^2 = \frac{1}{(\theta(r))^2}c^2dt^2 - (\theta(r))^2\left(dr^2 + r^2d\Omega^2\right) \qquad 0 \le r \le a$$

$$ds^2 = \frac{1}{\left(1-\frac{(a^2\theta'(a))}{R}\right)^2}c^2dT^2 - \left(1-\frac{(a^2\theta'(a))}{R}\right)^2\left(dR^2 + R^2d\Omega^2\right) \qquad A < R$$

where $A = \left(a\theta(a) + a^2\theta'(a)\right)$

References

1. Bonnor W.B and Wickramasuriya S.B.P, Int.J.Theor.Phys.5,371(1972).
2. Bonnor W.B and Wickramasuriya,Mon.Not.R.Astr.Soc.170,643(1975).
3. Wickramasuriya S.B.P PhD. Thesis, University of London.(1972).
4. R.Emden, Gaskugelin, Berlin and Leipzig(1907).