4.21 Coupling Shallow Water Equation with Navier-Stokes Equations: A viscous shallow water model.

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ABSTRACT

The general characteristic of shallow water flows is that the vertical characteristic scale D is essentially smaller than and typical horizontal scale L .i.e. $\varepsilon := \frac{D}{L} << 1$.
In many classical derivations, in order to obtain the shallow water approximation of the Navier-Stokes's Equations, the molecular viscosity effect is neglected and a posteriori is added into the shallow water model to represent the efficient-viscosity (a friction term through the Chezzy formula which involes empirical constants) at the bottom topography. However, the validity of this approach has been questioned in some applications as the models lead to different Rankine-Hugoniot curves (see e.g. [1]). Therefore, it can be useful to consider the molecular viscosity effect directly in the derivation of the shallow water model. On the other hand the classical shallow water models are derived under the assumption of slowly varying bottom topographies. Hence, for the description of incompressible shallow water laminar flow in a domain with a free boundary and highly varying bottom topography, the classical Shallow Water Equations are not applicable. The remedy consist of dividing the flow domain into two sub-domains namely, near field (sub domain with the bottom boundary) and far field (sub domain with the free boundary) with a slowly varying artificial interface and employ the Navier-Stokes Equations and Viscous Shallow Water Equations in the near field and far field, respectively.

In this work, we derive a two-dimensional Viscous Shallow Water model for incompressible laminar flows with free moving boundaries and slowly varying bottom topographies to employ in the far field. In this approach, the effect of the molecular viscosity is retained and thereby corrections to the velocities and the hydrostatic pressure approximations are established. Coupling modified shallow water model with NSE has been carried out in a separate work.

In order to derive the viscous shallow water model the two-dimensional Incompressible Navier-Stokes equations in usual notations

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}+\nu\frac{\partial w}{\partial z}\right),$$

$$\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g+\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial z}+\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial z}\left(2\nu\frac{\partial u}{\partial z}\right), \quad \text{---------------------------(1)}$$

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial z}=0.$$

are employed in the far field with the suitable boundary conditions. On the free surface, we assume that the fluid particle does not leave the free surface and we neglect the wind effect and the shear stress. On the artificial boundary we set the conditions according with the Navier-Stokes solution at the interface. On the lateral boundaries inflow and outflow conditions are employed. Rescaling the variables with the typical characteristic

scales L and D, the dimensionless form of the Navier-Stokes's equations for shallow water flows are obtained. Similarly, assuming that the bottom boundary is regular and the gradient of the free surface remains bounded we obtain the dimensionless boundary conditions. The second order terms with respect to ε in the system are neglected and asymptotic analysis is carried out under the assumptions, the flow quantities admit linear asymptotic expansion to the second order with respect to ε and the molecular viscosity of the water is very small. Then, rescaling the depth averaged first momentum equation of the resulting system and substituting the zeroth order solution for the velocity and the pressure in it the zeroth order first momentum equation which include the interface conditions is obtained. Again integrating the continuity equation of the dimensionless system from z_I to $H(t,x)$, a more detailed view of the vertical velocity component is established. Similarly, integrating the vertical momentum equation the dimensionless system from z_I to $H(t,x)$ and replacing boundary conditions, the second order correction to the hydrostatic pressure distribution is derived. Then, dropping $O(\varepsilon^2)$ in the system and switching to the variables with dimensions, the following results are established.

Proposition: *The formal second order asymptotic expansion of the Navier-Stokes Equations for the shallow water laminar flow is given by*

$$u(t,x,z) = u(t,x,z_I) + \left(1 - \frac{z - z_I}{2h}\right)\frac{\partial u}{\partial z}(x,z_I,t)(z - z_I) - \frac{1}{2h}\frac{\partial u}{\partial z}(x,z_I,t)(z - z_I)^2$$

$$w(t,x,z) = w(t,x,z_I) - \int_{z=z_I}^{h(t,x)+z_I} \frac{\partial u}{\partial x} d\eta$$

$$p(t,x,z) = g(h + z_I - z) - \nu\frac{\partial u}{\partial x}(t,x,z) - \nu\frac{\partial \bar{u}}{\partial x}(x,t)$$

with the viscous shallow water equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}h\right) = \left(w - u\frac{\partial z_I}{\partial x}\right)_{z=z_I},$$

$$\frac{\partial}{\partial t}\left(\bar{u}h\right) + \frac{\partial}{\partial x}\left(\bar{u}^2 h\right) + \frac{\partial}{\partial x}\left(\frac{gh^2}{2}\right) = \frac{\partial}{\partial x}\left(4\nu h\frac{\partial \bar{u}}{\partial x}\right) - \tau_I,$$

where $\tau_I = \left[p\frac{\partial z_I}{\partial x} + \nu\frac{\partial u}{\partial z} + \nu\frac{\partial w}{\partial x} - 2\nu\frac{\partial u}{\partial x}\frac{\partial z_I}{\partial x} + u\left(u\frac{\partial z_I}{\partial x} - w\right)\right]_{z=z_I}$ *and* $z = z_I(x,t)$ *is the interface.*

Concluding remarks

In the zeroth order expansion as well as in many classical shallow water models, the horizontal velocity does not change along with the vertical direction. In contrast, our first order correction gives a quadratic expansion to the vertical velocity components retaining more details of the flow. As many classical models we do not neglect the viscosity effect but just assume that it is very small. Also, the zeroth order hydrostatic pressure approximation has been upgraded to the first order giving a parabolic correction to the pressure distribution.

References

1. Miglio E., Quarteroni A., Salari F. *Coupling of free surface and ground water flows*, 2003, Computational Fluids, vol. 32, num. 1, p. 73-83
2. Wesseling P. *Principles of Computational Fluid Dynamics*, 2004, Springer International