4.20 Simple and analytical proof of Fermat's last theorem for n=3

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ABSTRACT

It is well known that Fermat's last Theorem, in general, is extremely difficult to prove although the meaning of the theorem is very simple. It is surprising that the proof of theorem for $n = 3$,the smallest, corresponding number, given by Leonard Euler, which has been recommended for amateurs[1], is not only difficult but also has a gap in the proof. Paulo Rebenboim claims[1] that he has patched up Euler's proof, which is very difficult to understand, however. It was shown[2] that the parametric solution for the x, y, z in the equation $z^3 = y^3 + x^3, (x, y) = 1$ could be obtained easily with one necessary condition that must be satisfied by the parameters. Fairly simple analytical proof of the Fermat's last theorem for $n = 3$ was given [2] using this necessary condition. In this contribution very much simpler proof is given ,which is very suitable for amateurs.

Fermat's last theorem for $n = 3$ can be stated as the equation

$$z^3 = y^3 + x^3,\ (x, y) = 1. \tag{1}$$

has no non-zero integral solution for (x, y, z). If we assume a non-zero($xyz \neq 0$) solution for (x, y, z) , then one of (x, y, z) is divisible by 3 . Since we can assume for (1) for negative integers, without loss of generality one can assume that y is divisible by 3. Then if $y = 3^{\beta}\alpha\gamma$, the parametric solution of (1) , can be expressed as

$$x = 3^{\beta}\alpha\theta\delta + \delta^3 \tag{a}$$

$$y = 3^{\beta}\alpha\theta\delta + 3^{3\beta-1}\alpha^3 \tag{b}$$

$$z = 3^{3\beta-1}\alpha^3 + 3^{\beta}\alpha\theta\delta + \delta^3 \tag{c}$$

and a necessary condition satisfied by the parameters is

$$\theta^3 - \delta^3 - 2.3^{\beta}\alpha\delta\theta - 3^{3\beta-1}\alpha^3 = 0 \tag{d}$$

In this equation ,θ is a factor of z ,δ is a factor of x and $\gamma = \theta\delta + 3^{2\beta-1}\alpha^2$.

Proof of the Fermat's last theorem for $n = 3$

Expressing $3^{3\beta-1}\alpha^3$ as $3^{3\beta-3}\alpha^3 + 8.3^{3\beta-3}$ and substituting $\theta = 3^{\beta-1}g + \delta$ in (d),one gets

$$(g - 2\alpha)(\delta^2 + 3^{\beta-1}g\delta + 3^{2\beta-3}(g^2 + 2\alpha g + 4\alpha^2)) = 3^{2\beta-3}\alpha^3 \tag{2}$$

the condition, $\theta = 3^{\beta-1}g + \delta$ is due to a simple lemma used in [2] from which it follows

that $\beta > 1$ .It is easy to deduce that $g-2\alpha$ is divisible by $3^{2\beta-3}$ since $(3,\delta)=1$ and $\beta>1$.
If $\alpha=\pm 1$, then , $g=\pm 2+3^{2\beta-3}$>0.
Now,

$$(\delta^2+3^{\beta-1}g\delta+3^{2\beta-3}(g^2 \pm 2\alpha g+4\alpha^2)=\pm 1 \tag{3}$$

is never satisfied since (2) can be expressed as

$$(\delta+\frac{3^{\beta-1}g}{2})^2+3^{2\beta-3}(\frac{g^2}{4}+2g+4)=\pm 1 \tag{4}$$

If $\alpha \neq \pm 1$, we deduces from (2) that $g-2\alpha=3^{2\beta-3}s^3$ where $s \neq 1$ and s is a factor of α. This is because factor of g cannot be a factor of both δ and α since $\theta=3^{\beta-1}g+\delta$ and α is a factor of y. Hence,

$$3^{2\beta-3}g^2+g(2\alpha 3^{2\beta-3}+3^{\beta-1}\delta)+\delta^2-q^3+4\alpha^2 3^{2\beta-3}=0 \tag{5}$$

where $\alpha=sq,(s,q)=1$. Now, this quadratic equation in g must be satisfied by $2\alpha+3^{2\beta-3}s^3$. If the roots of this quadratic are α_1 and α_2=$2\alpha+3^{2\beta-3}s^3$, then

$\alpha_1\alpha_2=\dfrac{\delta^2+4\alpha^2.3^{2\beta-3}-q^3}{3^{2\beta-3}}$, $\alpha_1+\alpha_2=-\dfrac{2\alpha.3^{2\beta-3}+3^{\beta-1}\delta}{3^{2\beta-3}}$ It is easy to obtain from these two relations that

$$3^{2\beta-3}\alpha_2=-(2\alpha.3^{2\beta-3}+3^{\beta-1}\delta)-\frac{(\delta^2+4.\alpha^2.3^{2\beta-3}-q^3)}{2\alpha+3^{2\beta-3}s^3} \tag{6}$$

Hence, $2\alpha+3^{2\beta-3}s^3$ is a factor of $\delta^2+4.\alpha^2.3^{2\beta-3}-q^3$. In other words, $2\alpha+3^{2\beta-3}s^3$ is a factor of the expression

$$(2\alpha)^3-(2\alpha)^2.8s^3 3^{2\beta-3}-8\delta^2 s^3 \tag{7}$$

or, $-3^{2\beta-3}s^3$ is an integral root of the equation

$$x^3-8s^3 3^{2\beta-3}x^2-8\delta^2 s^3=0 \tag{8}$$

This means that $3^{2\beta-3}s^3$ is a factor of $8\delta^2.s^3$ which contradicts $(3,\delta)=1$, that is $(3,x)=1$, and the proof of theorem is complete. It should be emphasized that the necessary condition needed for our proof can be obtained without obtaining the parametric solution of (1), making the proof given here much shorter.

References

(1).Paulo Ribenboim, Fermat's last theorem for amateurs, Springer-Verlag 1991,New York,INC

(2) Piyadasa R.A.D. Analytical proof of Fermat's last theorem for $n=3$,8th Annual research symposium 2007 pp 127