4.18 Exact formula for the sum of the squares of spherical Bessel and Neumann function of the same order

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ABSTRACT

The sum of the squares of the spherical Bessel and Neumann function of the same order (SSSBN)is the square of the modulus of the Hankel function when the argument of all function are real, and is very important in theoretical physics. However, there is no exact formula for SSSBN.Corresponding formula, which has been derived by G.N.Watson[1] is an approximate formula[1], [2] valid for $\mathrm{Re}(z) > 0$,and it can be written as $J_\nu^2(z)+N_\nu^2(z)\approx \frac{2}{\pi z}\sum_{k=0}^{\infty}\frac{(2k-1)!!\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k z^{2k}k!\Gamma\left(\nu-k+\frac{1}{2}\right)}$ and the error term R_p satisfies

$$|R_p| < \left|\frac{\cos\nu\pi}{\cos R(\nu\pi)}\right|\frac{p!|(R(\nu),p)|}{(2p)!}2^{2p}\sinh^{2p}t$$

where $\frac{\cosh 2\nu t}{\cosh t}=\sum_{m=0}^{p-1}\frac{m!(\nu,m)}{(2m)!}2^m\sinh^{2m}t+R_p$

Upper bound of R_p in the important case when $\nu = n+\frac{1}{2}$, is undefined since $\cos R(\nu\pi)=\cos\nu\pi = 0$,where R stands for the real part and $m!(\nu,m)=\frac{\Gamma(n+1+m)}{\Gamma(n+1-m)}$.

The same formula has been derived [1]by the method called Barne's method but the error tem is very difficult to calculate. In this contribution, we will show that an exact formula exists for SSSBN when the order of the Bessel and the Neumann function is $n+\frac{1}{2}$,and it can be written as $J^2_{n+\frac{1}{2}}(z)+N^2_{n+\frac{1}{2}}(z)=\frac{2}{\pi z}\sum_{k=0}^{n}\frac{(2k-1)!!\Gamma(n+k+1)}{2^k z^{2k}k!\Gamma(n-k+1)}$.

Proof of the formula

In order to show that the above formula is exact, one has to establish the identity,

$$\frac{\cosh(2n+1)t}{\cosh t}=\sum_{m=0}^{n}\frac{\Gamma(n+1+m)}{\Gamma(n+1-m)}\frac{2^{2m}\sinh^{2m}t}{2m!} \tag{1}$$

It is an easy task to show that the equation (1) holds for $n=0$ and $n=1$. Now, assume that the equation (1) is true for $n \le p$. It can be easily shown that

$$\cosh(2p+3)t = 4\cosh(2p+1)t.\sinh^2 t + 2\cosh(2p+1) - \cosh(2p-1)t \tag{2}$$

and hence the following formula holds.

$$\frac{\cosh(2p+3)t}{\cosh t}=\sum_{m=0}^{p}\frac{\Gamma(p+1+m)}{\Gamma(p+1-m)}\frac{2^{2m+2}\sinh^{2m+2}t}{2m!}+2\sum_{m=0}^{p}\frac{\Gamma(p+1+m)}{\Gamma(p+1-m)}\frac{2^{2m}\sinh^{2m}t}{2m!}-\sum_{m=0}^{p-1}\frac{\Gamma(p+m)}{\Gamma(p-m)}\frac{2^{2m}\sinh^{2m}t}{2m!}$$

$$= 1 + \overbrace{\sum_{m=1}^{p-1} \frac{\Gamma(p+m)}{\Gamma(p-m+2)} \frac{2^{2m}\sinh^{2m} t}{(2m-2)!} + \sum_{m=1}^{p-1} \frac{\Gamma(p+1+m)}{\Gamma(p+1-m)} \frac{2^{2m}\sinh^{2m} t}{2m!} + \sum_{m=1}^{p-1} \frac{\Gamma(p+m)}{\Gamma(p-m+1)} \frac{2^{2m}\sinh^{2m} t}{(2m-1)!}}^{Q}$$

$$+ P$$

where $p = \dfrac{2\Gamma(2p+1))2^{2p}\sinh^{2p} t}{2p!} + \dfrac{\Gamma(2p)}{\Gamma(2)} \dfrac{2^{2p}\sinh^{2p} t}{2(p-1)!} + \Gamma(2p+1)\dfrac{2^{2(p+1)}\sinh^{2(p+1)} t}{2p!}$

It can be shown that

$$Q = \sum_{m=1}^{p-1} 2^{2m}\sinh^{2m} t \frac{(p+m+1)!}{(p-m+1)!(2m)!} \quad \text{and } P = (2p+1)2^{2p}\sinh^{2p} t + 2^{2(p+1)}\sinh^{2(p+1)} t$$

Hence , $\dfrac{\cosh(2p+3)t}{\cosh t} = \displaystyle\sum_{s=0}^{p+1} \frac{\Gamma(p+m+2)}{\Gamma(p+2-m)} \frac{2m\sinh^{2m} t}{2(m)!}$

Since (1) is now true for $n \le p+1$, by the mathematical induction , the equation (1) is true for all n.By Nicholson's formula[1],

$$J_\nu^{\,2}(z) + N_\nu^{\,2}(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t\,dt \tag{3}$$

where $K_0(z) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} e^{-z\cosh t} dt$ is the modified Bessel function of the second kind of the

zero order. Substituting for $\cosh(2\nu.t)$ from (1) and using

$$\int_0^\infty K_0(t) t^{\mu-1} dt = 2^{\mu-2}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu}{2}\right) \tag{4}$$

we obtain $J_{n+\frac{1}{2}}^{\;2}(z) + N_{n+\frac{1}{2}}^{\;2}(z) = \dfrac{2}{\pi z}\displaystyle\sum_{k=0}^{n} \frac{(2k-1)!!\Gamma\left(\nu+k+\frac{1}{2}\right)}{2^k z^{2k} k!\Gamma\left(\nu-k+\frac{1}{2}\right)}$

from which the square of the modulus of the Hankel function follows immediately.

References

(1) Watson G.N. (1944), Treatise on the theory of Bessel functions 2nd ed .pp 441-448 Cambridge University Press

(2).Gradshtegn I.S & Ryzhik I.M.,(1980).Table of integrals,series and products, pp965 Academic press,INC