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Paper: Transformability

Cosmological constant in gravitational lensing

Consider the Schwarzschild de Sitter Metric,

$$ds^2 = \left(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \qquad (1)$$

The constant term $\frac{2GM}{c^2}$ is recognized as the Schwarzschild radius (r_s), and typically it is replaced by a constant term $2m$, where $m = \frac{1}{2} r_s = \frac{GM}{c^2}$ and then the equation (1) can be written as follows.

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2). \qquad (2)$$

Λ is the cosmological constant.

The null-geodesic equation in Schwarzschild-de Sitter metric can be written as,

$$\frac{E^2}{c^2} - l^2 u'^2 - l^2 u^2 + 2ml^2 u^3 + \frac{\Lambda l^2}{3} = 0 \text{, [1]} \qquad (3)$$

where E is the energy, l is the orbital angular momentum, Λ is the cosmological constant, $u = \frac{1}{r}$ and $u' = \frac{du}{d\phi}$.

Differentiating (3) with respect to ϕ,

$$u'(u'' + u - 3mu^2) = 0. \qquad (4)$$

Neglecting the solution, $u' = 0$ which implies u = *constant*, the equation of a light ray trajectory can be written as,

$$u'' + u = 3mu^2. \qquad (5)$$

The zeroth order solution and the first order solution of the equation (5) that represent the light ray trajectory are respectively given below.

$$u_0 = \frac{1}{r_0} \cos\phi \text{ [2]}, \qquad (6)$$

$$u = \frac{1}{r_0} \cos\phi - \frac{\varepsilon}{3r_0^2} \cos^2\phi + \frac{2\varepsilon}{3r_0^2} \text{ [2]}, \qquad (7)$$

where $\varepsilon = 3m$.

In general, in the literature, it is assumed that (7) is a solution of equation (3) without considering the limitations imposed. In this paper we discuss conditions under which (7) is a solution of equation (3).

Now the orbital angular momentum, $l = pr_0$ where p is the linear momentum.

The linear momentum, $p = \frac{E}{c}$.

Therefore,

$$l = \frac{E}{c} r_0. \tag{8}$$

Substituting (7) and (8) in (3), we have,

$$\begin{aligned} &\frac{E^2}{c^2} - l^2\left[-\frac{1}{r_0}\sin\phi + \frac{2\varepsilon}{3r_0^{\,2}}\sin\phi\cos\phi\right]^2 - l^2\left[\frac{1}{r_0}\cos\phi - \frac{\varepsilon}{3r_0^{\,2}}\cos^2\phi + \frac{2\varepsilon}{3r_0^{\,2}}\right]^2 \\ &+ \frac{2\varepsilon}{3} l^2\left[\frac{1}{r_0}\cos\phi - \frac{\varepsilon}{3r_0^{\,2}}\cos^2\phi + \frac{2\varepsilon}{3r_0^{\,2}}\right]^3 + \frac{\Lambda l^2}{3} = 0. \end{aligned} \tag{9}$$

By simplifying the above equation and since $l \neq 0$ we obtain the following equation,

$$2m\left[\begin{aligned} &\frac{8\varepsilon^3}{27r_0^{\,6}} - \frac{4\varepsilon^3}{9r_0^{\,6}}\cos^2\phi + \frac{2\varepsilon^3}{9r_0^{\,6}}\cos^4\phi - \frac{\varepsilon^3}{27r_0^{\,6}}\cos^6\phi + \frac{\varepsilon^2}{3r_0^{\,5}}\cos^5\phi - \frac{4\varepsilon^2}{3r_0^{\,5}}\cos^3\phi + \frac{4\varepsilon^2}{3r_0^{\,5}}\cos\phi \\ &-\frac{2\varepsilon}{3r_0^{\,4}} + \frac{2\varepsilon}{r_0^{\,4}}\cos^2\phi - \frac{\varepsilon}{2r_0^{\,4}}\cos^4\phi \end{aligned}\right] + \frac{\Lambda}{3} = 0$$

$$\Lambda = -18m^2\left[\begin{aligned} &\frac{8m^2}{3r_0^{\,6}} - \frac{4m^2}{r_0^{\,6}}\cos^2\phi + \frac{2m^2}{r_0^{\,6}}\cos^4\phi - \frac{m^2}{3r_0^{\,6}}\cos^6\phi + \frac{m}{r_0^{\,5}}\cos^5\phi \\ &-\frac{4m}{r_0^{\,5}}\cos^3\phi + \frac{4m}{r_0^{\,5}}\cos\phi - \frac{2}{3r_0^{\,4}} + \frac{2}{r_0^{\,4}}\cos^2\phi - \frac{1}{2r_0^{\,4}}\cos^4\phi \end{aligned}\right]. \tag{10}$$

From (10) it is clear that the solution given by (7) of equation (3) is valid only if Λ is a constant of order m^2, and as we neglect terms of order 2 and above we are justified in assuming (7) as a solution of equation (3). However, it turns out that this particular solution is valid only if Λ is a constant of order 2 or more in *m*. If Λ is a non zero constant and of order one in *m*, the solution (7) is not valid and we have to seek other solutions.

References

[1] Jayakody, J.A.N.K., de Silva, L.N.K. *'Path of a light ray near a body with cosmological constant'*, 10th Annual Research Symposium 2009, University of Kelaniya (2009).

[2] Adler, R., Bazin, M., Schiffer, M., (1965) *Introduction to General Relativity*, United States of America: McGraw-Hill, Inc.