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### New Set of Primitive Pythagorean Triples and Proofs of Two Fermat's Theorems

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#### Extended Abstract

##### 1. Introduction

Fermat used the well known primitive Pythagorean set [1], [2],

$$x = 2ab, y = a^2 - b^2, z = a^2 + b^2 \quad (1.1)$$

, where  $a > b > 0$  and  $a, b$  are of opposite parity  $(a, b) = 1$ , to prove the two theorems

(1) Fermat's last theorem for  $n = 4$

(2) the area of a Pythagorean triangle cannot be a square of an integer.

Historically the above set was well known long ago and no other primitive whole set was available in the literature. However, the following complete primitive set can be obtained from the Pythagoras' equation easily.

The set

$$x = ab, y = \frac{1}{2}(a^2 - b^2), z = \frac{1}{2}(a^2 + b^2) \quad (1.2)$$

where  $a > b > 0$  and  $a, b$  are both odd and  $(a, b) = 1$  can be obtained from the Pythagoras equation

$$z^2 = y^2 + x^2, (x, y) = 1 \quad (1.3)$$

To obtain this set, assume that  $x$  is odd and  $y$  is even. Obviously,  $z$  is odd.

Then  $z^2 - y^2 = x^2, (z - y)(z + y) = x^2$ , and  $(z - y), (z + y)$  are co-prime.

Hence  $z + y = a^2, z - y = b^2$ , where  $x = ab, a > b > 0$  and both  $a, b$  are odd.

Now,  $x = ab, y = \frac{1}{2}(a^2 - b^2), z = \frac{1}{2}(a^2 + b^2)$ , where  $a > b > 0$  and both  $a, b$  are odd and co-prime, give the complete primitive set of Pythagorean triples.

##### 2. Proof of Two Theorems

(1) Fermat's last theorem for  $n = 4$  can be stated as there are no non-trivial integral triples

$(x, y, z)$  satisfying the equation

$$z^4 = y^4 + x^4, (x, y) = 1 \text{ where } z \text{ is odd.} \quad (2.1)$$

This can be stated as there is no non-trivial integral triples  $(x, y, z)$  satisfying the equation:

$$z^4 - x^4 = (y^2)^2 = c^2.$$

- (2) The second theorem can be stated as there are no integers  $x, y, c$  satisfying the equation

$$\frac{1}{2}xy = c^2, \text{ where } x^2 + y^2 = z^2, (x, y) = 1. \quad (2.2)$$

In terms of the new Pythagorean triples this can be stated as

$ab(a^2 - b^2) = 4c^2 = d^2$ , where  $a, b$  are odd. In other words, there are integers  $m, n, q$  satisfying the equation

$$m^4 - n^4 = q^2 \quad (2.3)$$

where we have used the fact that  $a, b$  are squares.

Now, we will show that there are no non-trivial integer triples satisfying the equation

$$z_1^4 - x_1^4 = y_1^2 \quad (2.4)$$

using the new set of Pythagorean triples, in order to prove the two theorems.

In the following, we prove the two theorems using the method available in the literature [1],

[2] but using the new set of Pythagorean triples:

$$z_1^2 = \frac{1}{2}(a^2 + b^2), \quad x_1^2 = ab, \quad \text{and} \quad y_1 = \frac{1}{2}(a^2 - b^2)$$

$$\text{If } t_1 = \frac{1}{2}(a - b), \quad t_2 = \frac{1}{2}(a + b),$$

$$\text{then } z_1^2 = t_1^2 + t_2^2, \quad x_1^2 = t_2^2 - t_1^2$$

Assume that  $z_1$  is the smallest integer satisfying the equation (2.4).

Now, we have  $t_2^4 - t_1^4 = z^2 x^2 = q_1^2$ , where  $z_1 > t_2$ .

Therefore, we deduce by the method of infinite descent that there are no non-trivial integer triples satisfying the equations (2.4) or (2.3). This completes the proof of the two theorems.

## References

- [1] P. Ribenboim, Fermat's last theorem for amateurs, Springer-Verlag, New York, 1991  
 [2] H.M. Edwards, Fermat's last theorem, A Genetic Introduction to Algebraic Number Theory, Springer - Verlag, 1977.